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# Complex Path Integral Representation for Semiclassical Linear Functionals

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Semiclassical linear functionals are characterized by the distributional equation $D(\phi L)+\psi L=0$ where $\phi$ and $\psi$ are arbitrary polynomials with the condition $\deg(\psi)\geqslant 1$. Two cases are considered:

(A) $\deg(\phi)>\deg(\psi)$

(B) $\deg(\phi)\leqslant\deg(\psi)$.

In an earlier work by the authors (*J. Comput. Appl. Math.* **57** (1995), 239–249) integral representations are given for semiclassical functionals in case (A). Here the problem is continued and case (B) is solved: it is always possible to choose some path $\gamma$ in the complex plane such that every solution, regular or not, of $D(\phi L)+\psi L=0$ can be represented in the form $\langle L, p\rangle=\int_\gamma w(z)\,p(z)\,dz$ where $w(z)$ is a solution of the differential equation $(\phi w)'+\psi w=0$. In some cases, the expression for $L$ is a singular integral and a regularization process is given. 

## 1. INTRODUCTION

The first authors to study semiclassical orthogonal polynomials were E. N. Laguerre [8] and J. Shohat [18]. Recently, a unified theory of these polynomials has been developed by P. Maroni in [11, 13, 14], where the distributional equation defines the moment functional associated with the semiclassical orthogonal polynomials. This equation is the starting point in this paper.



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DEFINITION 1.1. A regular moment functional $L$ is said to be semiclassical if and only if there exist polynomials $\phi$ and $\psi$, $\deg(\psi) \geqslant 1$ such that

$$D(\phi L) + \psi L = 0. \tag{1}$$

Given $L$, among the pairs $(\phi, \psi)$ which satisfy (1) let $s$ be the minimum of $\max\{\deg(\phi) - 2, \deg(\psi) - 1\}$. Then $L$ is said to be of class $s$. (If $s = 0$, the regular solutions of (1) are the functionals corresponding to the classical polynomials.)

As usual, $\langle DL, p \rangle = -\langle L, p' \rangle$ and $\langle \phi L, p \rangle = \langle L, \phi p \rangle$.

As regards the problem of the integral representation, the classical case was solved by J. L. Geronimus [4], by R. D. Morton and A. M. Krall [16], and by M. E. H. Ismail *et al.* [7]. For $s > 0$, examples have been given in [2, 5, 6]; the whole class of semiclassical functionals which are positive definite on the real line is given in [3]. A. P. Magnus in [9] solved the problem for "generic semiclassical" orthogonal polynomials which correspond to regular solutions of (1) in case (A) provided that the zeros of the polynomial $\phi$ are distinct. In [10], a solution for the problem in case (A) without restrictions on $\phi$ was given and the problem in case (B) was started. We summarize these results:

PROPOSITION 1.1. (a) *If $L$ is an (A)-functional and $\mu_n$, $n \geqslant 0$, its moments, then there exist a positive integer $N$ and positive constants $C$ and $M$ such that*

$$|\mu_n| \leqslant CM^n, \qquad n \geqslant N.$$

(b) *If $L$ is an (B)-functional, there exist a positive integer $N$ and positive constants $C$ and $M$ such that*[1]

$$|\mu_n| \leqslant CM^n n!, \qquad n \geqslant N.$$

From (a), the Stieltjes function associated with an (A)-functional, $S(z) = -\sum_{n=0}^{\infty} (\mu_n / z^{n+1})$, is an analytic function in $|z| > M$ and $L$ can be represented in the form

$$\langle L, p \rangle = \frac{1}{2\pi i} \int_{|z| = M^*} p(z)\, w(z)\, dz, \qquad M^* > M.$$

The form of $S(z)$ is obtained from its differential equation $(\phi S)' + \psi S = D$, where $D(z)$ is a polynomial of degree $s$. This is a characteristic of the semiclassical functionals (see [14]).

[1] From Carleman's criteria it follows that, for positive definite functionals on the real line, the problem of moments is determined.

For (B)-functionals, $\phi(x)=\sum_{k=0}^{s+1} a_k x^k$ and $\psi(x)=\sum_{k=0}^{s+1} b_k x^k$ with $b_{s+1}\neq 0$ and, since Eq. (1) is equivalent to the fact that the moments $\mu_n$ of $L$ satisfy

$$-n\sum_{k=0}^{s+1} a_k \mu_{n+k-1}+\sum_{k=0}^{s+1} b_k \mu_{n+k}=0, \qquad n=0, 1, ...,$$

the set of solutions is a linear space of dimension $s+1$. A basis of this space will be called a Fundamental System of Solutions (FSS).

Let $w(z)$ be a function and $\gamma$ a path in the complex plane such that

$$(\phi(z)\, w(z))' + \psi(z)\, w(z)=0, \tag{2}$$

$$\phi(z)\, w(z)\, p(z)|_{\gamma}=0 \qquad \text{for every polynomial } p. \tag{3}$$

The moment functional $L$ defined by

$$\langle L, p\rangle = \int_{\gamma} p(z)\, w(z)\, dz \tag{4}$$

is a solution of (1) because

$$\langle D(\phi L)+\psi L, p\rangle = \int_{\gamma} (-\phi(z)\, w(z)\, p'(z)+\psi(z)\, w(z)\, p(z))\, dz$$

and, by integration by parts, this is the same as

$$-\phi(z)\, w(z)\, p(z)|_{\gamma}+\int_{\gamma} ((\phi(z)\, w(z))' + \psi(z)\, w(z))\, p(z)\, dz=0$$

from-conditions (2) and (3). This technique was described by L. M. Milne-Thomson in [15].

In [10] it has been proved that it is possible to find $s+1$ independent solutions of (1), provided that $\phi=1$ and $\psi$ is an arbitrary polynomial of degree $\geqslant 1$, in the form (4) such that conditions (2) and (3) hold. Next, the same will be proved for the general case (B).

## 2. INTEGRAL REPRESENTATION OF (B)-FUNCTIONALS

Since the problem in case $\phi=1$ is solved in [10], here the polynomial $\phi$ is always considered to have some zero. After a linear change in the variable (see [14, Proposition 6.2]) the type (B) equation may be written in such a way that one of the roots is zero and the leading coefficient of $\psi$ is an appropriate number which simplifies calculations

$$D(\phi L)+\psi L=0$$

$$\phi(z)=z^{r_0+1}\prod_{k=1}^{M}(z-a_k)^{r_k+1}, \qquad \deg(\phi)=\sum_{k=0}^{M}(r_k+1)=N+1\leqslant s+1. \qquad (5)$$

$$\psi(z)=(s-N+1)\,z^{s+1}+\cdots.$$

If some $r_k>0$, we suppose that $r_0>0$. Of course, $M$ may be zero. Solving the differential equation of condition (2), $(\phi w)'+\psi w=0$, we obtain

$$w(z)=z^{\alpha_0}\sum_{k=1}^{M}(z-a_k)^{\alpha_k}\exp(-z^{s-N+1}\cdots)$$
$$\times\exp\left(\frac{A_0}{z^{r_0}}+\sum_{k=1}^{M}\frac{A_k}{(z-a_k)^{r_k}}\right)\exp\left(\frac{Q(z)}{R(z)}\right), \qquad (6)$$

where $Q(z)$ and $R(z)$ are polynomials with $\deg Q(z)<\deg R(z)$ and such that, in the decomposition of $Q(z)/R(z)$ in partial fractions, the exponent of each term corresponding to the zero $a_k$ is less than $r_k$.

*We impose the following restrictions*:

- $\phi$ and $\psi$ do not have any common zero. (With Proposition 3.2 in the next section, the problem in the general case is solved.)
- If some $a_k$ is a simple zero, the corresponding exponent $\alpha_k$ is such that $\Re\alpha_k>-1$. (The other possibility will be considered in the next section.)

In order to simplify notation, we finally suppose that $A_k=-1$, $k=0, ..., M$. An appropriate rotation around each zero $a_k$ for the paths $\Gamma_{k,j}$, defined below, can be choosen which enables one to solve the equation when $A_k\neq-1$.

*Now we define the paths* such that condition (3) holds.

For each zero $a_k$, $k=0, ..., M$, which is a multiple zero and for $j=1, ..., r_k$, we define:

- $\beta_{k,j}$ the $r_k$-roots of the unity. We also consider $\arg(\beta_{k,r_k+1})=2\pi$.
- $l_k$ is a positive real number whose length will be defined later.
- $\gamma_{k,j}$ is the segment from $a_k$ in the direction $\beta_{k,j}$ and length $l_k$.
- $C_{k,j}$ is the arc of the circumference of radius $l_k$, centered on $a_k$, which extends from $\arg(\beta_{k,j})$ until $\arg(\beta_{k,j+1})$.

For each $k$, we define the length $l_k$ to be small enough for the arcs $C_{k,j}$ of different zeros not to have any point in common.

*The paths from* $a_0 = 0$ *to* $a_k$, $k = 1, ..., M$:

- $E_k$ is any simple curve beginning at the origin and extending in some direction $d_k$ such that, when $z \in d_k$, $\lim_{z \to 0} \exp(-1/z^{r_0}) = 0$, arriving at $a_k$ in direction $\beta_{k,1}$ when $a_k$ is a multiple zero or in any direction when $a_k$ is a simple zero, and in such a way that, avoiding points $a_j$, $j \neq k$, two different $E_k$ have only the origin in common.

Finally, for $m = 1, ..., s - N + 1$, *the paths joining zero and infinity*:

- $\beta_m$ are the $s - N + 1$-roots of the unity.
- $l_0^*$ is a positive real number such that every path $\gamma_{k,j}$ and $E_k$ is inside the disk centered on the origin and with radius $l_0^*$.
- $R_0$ is an arc joining 0 and $l_0^*$ along the real line and avoiding points $a_k$ if any of them is a positive real number.
- $C_m$ is the arc of the circumference centered on the origin and radius $l_0^*$ such that it goes from zero argument to the argument of $\beta_m$.
- $R_m$ is the line in the direction of $\beta_m$ corresponding to $l_0^* \leqslant |z| < \infty$.

Then, let

$$\Gamma_m = R_0 \cup C_m \cup R_m, \qquad m = 1, ..., s - N + 1,$$

$$\Gamma_{k,j} = \gamma_{k,j} \cup C_{k,j} \cup (-\gamma_{k,j+1}), \qquad j = 1, ..., r_k, \quad k = 0, ..., M,$$

(see Fig. 1) and the corresponding functionals

$$\langle L_m, p \rangle = \int_{\Gamma_m} p(z)\, w(z)\, dz, \qquad m = 1, ..., s - N + 1 \tag{7}$$

$$\langle L_{k,j}, p \rangle = \int_{\Gamma_{k,j}} p(z)\, w(z)\, dz, \qquad j = 1, ..., r_k \text{ for each } k \text{ such that } r_k > 0, \tag{8}$$

$$\langle L_k^*, p \rangle = \int_{E_k} p(z)\, w(z)\, dz, \qquad k = 1, ..., M. \tag{9}$$

Thus we have $s - N + 1 + r_0 + \cdots + r_M + M = s + 1$ solutions of Eq. (5).

THEOREM 2.1. $\{L_1, ..., L_{s-N+1}, L_{0,1}, ..., L_{0,r_0}, ..., L_{M,1}, ..., L_{M,r_M}, L_1^*, ..., L_M^*\}$ *is an FSS of Eq.* (5).

We have to prove that they are independent functionals and we begin the proof with two auxiliary results. The first one is straightforward.

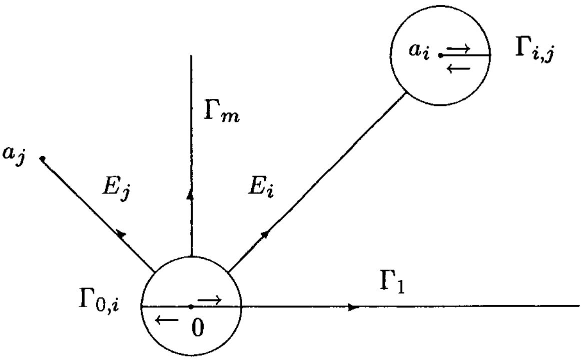


FIGURE 1

LEMMA 2.1. *Let $D(\phi L)+\psi L=0$ be a type* (B) *equation of class s. A set of solutions $\{L_1, ..., L_{s+1}\}$ is an FSS if and only if*

$$\det(\langle L_i, (x-a)^n\rangle)_{i=1, n=0}^{s+1, s} \neq 0$$

*for any complex number a.*

The following lemma is a kind of Theorem of Final-value for the Laplace transform (see [19, p. 249]).

LEMMA 2.2. *Let $q(x)=-x^n+\sum_{k=1}^n b_k x^{n-k}$ where $b_k \in \mathscr{C}$. Let $f(x)$ be a locally integrable bounded function in $[0, \infty)$. Let $H(\alpha)$ be the function*

$$H(\alpha)=\int_0^\infty x^\alpha \exp(q(x))\, dx, \qquad \Re(\alpha)>-1$$

*and, for every fixed α, let $F(t)$ be the function*

$$F(t)=\int_0^\infty x^\alpha \exp(q(tx))\, f(x)\, dx, \qquad t>0.$$

*If $\lim_{x\to\infty} f(x)=A$ then $\lim_{t\to 0^+} t^{\alpha+1}F(t)=AH(\alpha)$.*

*Proof.*

$$H(\alpha)=\int_0^\infty x^\alpha \exp(q(x))\, dx=\int_0^\infty (tx)^\alpha \exp(q(tx))\, t\, dx, \qquad t>0.$$

For a given $\varepsilon > 0$, let $T$ be such that $|f(x) - A| < \varepsilon$ for $x > T$. Then

$$|t^{\alpha+1}F(t) - AH(\alpha)| = \left| t^{\alpha+1} \int_0^\infty x^\alpha \exp(q(tx))(f(x) - A)\,dx \right|$$

$$\leqslant |t^{\alpha+1}| \int_0^T |x^\alpha \exp(q(tx))|\,|f(x) - A|\,dx$$

$$+ \varepsilon \int_T^\infty |x^\alpha \exp(q(tx))\,t^{\alpha+1}|\,dx$$

$$\leqslant |t^{\alpha+1}|\,TM + \varepsilon \int_0^\infty |x^\alpha \exp(q(x))|\,dx,$$

where $M$ is an upper bound of the function $|x^\alpha \exp(q(tx))|\,|f(x) - A|$ for $x \in [0, T]$ and $t \in [0, t_0]$ for some fixed $t_0$. Hence

$$\lim_{t \to 0^+} |t^{\alpha+1}F(t) - AH(\alpha)| \leqslant \varepsilon \int_0^\infty |x^\alpha \exp(q(x))|\,dx$$

and $\lim_{t \to 0^+} t^{\alpha+1}F(t) = AH(\alpha)$ follows. ∎

*Proof of the Theorem.* (I) First, we consider the particular case

$$D(xL) + ((s+1)\,x^{s+1} + \cdots)\,L = 0.$$

The only paths now are from zero to infinity, so we simplify notation

$$\langle L_j, p \rangle = \int_{\gamma_j} p(z)\,w(z)\,dz, \qquad j = 1, ..., s+1,$$

where

$$\gamma_j \equiv \beta_j x, \qquad 0 \leqslant x < \infty, \qquad \beta_j^{s+1} = 1, \qquad \text{and}$$

$$w(z) = z^\alpha \exp(-z^{s+1} + q(z)), \qquad \deg q \leqslant s.$$

If $\sum_{j=1}^{s+1} \lambda_j L_j = 0$ then

$$\left\langle \sum_{j=1}^{s+1} \lambda_j L_j, z^{n(s+1)+k} \right\rangle = 0; \qquad k = 0, ..., s; \quad n = 0, 1, ... . \tag{10}$$

Since

$$\langle L_j, z^{n(s+1)+k} \rangle$$

$$= \int_0^\infty x^{n(s+1)+k+\alpha} \beta_j^{\alpha+k+1} \exp(-x^{s+1} + q(\beta_j x))\, dx$$

$$= \frac{1}{s+1} \int_0^\infty \exp(-t)\, t^{n+(k+\alpha-s)/(s+1)} \beta_j^{\alpha+k+1} \exp(q(\beta_j t^{1/(s+1)}))\, dt,$$

for each fixed $k$, (10) becomes

$$0 = \frac{1}{s+1} \int_0^\infty \exp(-t)\, t^{n+(k+\alpha-s)/(s+1)} \sum_{j=1}^{s+1} \lambda_j \beta_j^{\alpha+k+1} \exp(q(\beta_j t^{1/(s+1)}))\, dt$$

$$= \frac{1}{s+1} (-1)^n F_k^{(n)}(1), \qquad n = 0, 1, ...,$$

where $F_k(y)$ is the Laplace transform of

$$t^{(k+\alpha-s)/(s+1)} \sum_{j=1}^{s+1} \lambda_j \beta_j^{\alpha+k+1} \exp(q(\beta_j t^{1/(s+1)})).$$

As a consequence $F_k(y) = 0$ and

$$\sum_{j=1}^{s+1} \lambda_j \beta_j^{\alpha+k+1} \exp(q(\beta_j t^{1/(s+1)})) = 0, \qquad k = 0, ..., s$$

follows. Since $\det(\beta_j^{\alpha+k+1})_{j=1, k=0}^{s+1, s} \neq 0$, we have

$$\lambda_j \exp(q(\beta_j t^{1/(s+1)})) = 0, \qquad j = 1, ..., s+1$$

from which $\lambda_j = 0$, $j = 1, ..., s+1$, and $\{L_1, ..., L_{s+1}\}$ is an FSS.

(II) *General case.* Now we write $w(z) = z^{\alpha_0} \prod_{k=1}^{M} (z - a_k)^{\alpha_k} \times \exp(q(z))\, f(z)$ and suppose

$$\sum_{m=1}^{s-N+1} \lambda_m L_m + \sum_{k=0}^{M} \sum_{j=1}^{r_k} \lambda_{k,j} L_{k,j} + \sum_{k=1}^{M} \lambda_k^* L_k^* = 0. \tag{11}$$

Then

$$\sum_{m=1}^{s-N+1} \lambda_m \int_{R_m} w(z)\, p(z)\, dz$$
$$= -\sum_{k=0}^{M} \sum_{j=1}^{r_k} \lambda_{k,j} \int_{\Gamma_{k,j}} p(z)\, w(z)\, dz - \sum_{k=1}^{M} \lambda_k^* \int_{E_k} p(z)\, w(z)\, dz$$
$$- \sum_{m=1}^{s-N+1} \lambda_m \int_{R_0 \cup C_m} p(z)\, w(z)\, dz \qquad \text{for every polynomial } p(z). \tag{12}$$

Let $\mu$ be a positive integer such that $\Re(\alpha_0 + \cdots + \alpha_k + \mu) > -1$, let

$$F_p(t) = \sum_{m=1}^{s-N+1} \lambda_m \int_{R_m} z^{p+\alpha_0+\mu} \prod_{k=1}^{M} (z-a_k)^{\alpha_k} \exp(q(tz))\, f(z)\, dz$$
$$= \sum_{m=1}^{s-N+1} \lambda_m \int_{R_m} z^{p+\alpha_0+\cdots+\alpha_M+\mu}$$
$$\times \prod_{k=1}^{m} \left(1 - \frac{a_k}{z}\right)^{\alpha_k} \exp(q(tz))\, f(z)\, dz, \qquad p = 0, ..., s-N,$$

and

$$G_p(t) = -\sum_{k=0}^{M} \sum_{j=1}^{r_k} \lambda_{k,j} \int_{\Gamma_{k,j}} z^{p+\alpha_0+\mu} \prod_{k=1}^{M} (z-a_k)^{\alpha_k} \exp(q(tz))\, f(z)\, dz$$
$$- \sum_{k=1}^{M} \lambda_k^* \int_{E_k} z^{p+\alpha_0+\mu} \prod_{k=1}^{M} (z-a_k)^{\alpha_k} \exp(q(tz))\, f(z)\, dz$$
$$- \sum_{m=1}^{s-N+1} \lambda_m \int_{R_0 \cup C_m} z^{p+\alpha_0+\mu} \prod_{k=1}^{M} (z-a_k)^{\alpha_k} \exp(q(tz))\, f(z)\, dz,$$
$$p = 0, ..., s-N.$$

$F_p(t)$ is an analytic function in $\Re(t^{s-N+1}) > 0$ and so is $G_p(t)$ in the whole complex plane. From (12), $F_p^{(n)}(1) = G_p^{(n)}(1)$, $n = 0, 1, ...$ and $F_p(t) = G_p(t)$, $t \in \Re(t^{s-N+1}) > 0$, follows for $p = 0, ..., s-N$. As a consequence

$$\lim_{t \to 0^+} t^{\alpha_0+\cdots+\alpha_M+\mu+p+1} F_p(t)$$
$$= \lim_{t \to 0^+} t^{\alpha_0+\cdots+\alpha_M+\mu+p+1} G_p(t) = 0, \qquad p = 0, ..., s-N.$$

Moreover,

$$F_p(t) = \sum_{m=1}^{s-N+1} \lambda_m \int_{l_0^*}^{\infty} (\beta_m x)^{p+\alpha_0+\cdots+\alpha_M+\mu} \times \prod_{k=1}^{M} \left(1 - \frac{a_k}{\beta_m x}\right)^{\alpha_k} \exp(q(t\beta_m x))\, f(\beta_m x)\, \beta_m\, dx$$

$$= \sum_{m=1}^{s-N+1} \lambda_m \int_{0}^{\infty} (\beta_m x)^{p+\alpha_0+\cdots+\alpha_M+\mu} \times \prod_{k=1}^{M} \left(1 - \frac{a_k}{\beta_m x}\right)^{\alpha_k} \exp(q(t\beta_m x))\, f(\beta_m x)\, \beta_m \chi_{[l_0^*, \infty]}(x)\, dx,$$

where $\chi_{[l_0^*, \infty)}(x) = 1$ when $x \in [l_0^*, \infty)$ and zero otherwise.

Since $\exp(\beta_m x) = \exp(-x^{s-N+1} + \cdots)$, we can apply Lemma 2.2 to each term in the expression of $F_p(t)$ and obtain

$$0 = \lim_{t \to 0^+} t^{\alpha_0+\cdots+\alpha_M+\mu+p+1} F_p(t)$$

$$= \sum_{m=1}^{s-N+1} \lambda_m \beta_m^{\alpha_0+\cdots+\alpha_M+\mu+p+1} \int_0^{\infty} x^{\alpha_0+\cdots+\alpha_M+\mu+p} \exp(q(\beta_m x))\, dx$$

for $p = 0, ..., s-N$, because $\lim_{x \to \infty} \prod_{k=1}^{M} (1-(a_k/\beta_m x))^{\alpha_k} f(\beta_m x) = 1$, $m = 1, ..., s-N+1$. This system is the same as

$$0 = \sum_{m=1}^{s-N+1} \lambda_m \int_{\delta_m} z^{\alpha_0+\cdots+\alpha_M+\mu+p} \exp(q(z))\, dz, \qquad p = 0, ..., s-N,$$

where $\delta_m$ is the line $z = \beta_m x$, $0 \leqslant x < \infty$, and its determinant may be written in the form

$$\det(\langle \bar{L}_m, z^p \rangle)_{m=1,\, p=0}^{s-N+1,\, s-N},$$

where $\{\bar{L}_1, ..., \bar{L}_{s-N+1}\}$ is an FSS for

$$D(xL) - (xq'(x) + \alpha + 1)\, L = 0, \qquad \alpha = \alpha_0 + \cdots + \alpha_M + \mu,$$

as was proved for the particular case of part (I). Taking into account Lemma 2.1, this determinant is non-zero and $\lambda_m = 0$, $m = 1, ..., s-N+1$ follows.

Equation (11) becomes

$$\sum_{k=0}^{M} \sum_{j=1}^{r_k} \lambda_{k,j} L_{k,j} + \sum_{k=1}^{M} \lambda_k^* L_k^* = 0. \tag{13}$$

Suppose that some $a_k$, and therefore $a_0$ $(=0)$ is a multiple zero; otherwise there is nothing to prove with respect to $\lambda_{k,j}$. It will be proved that $\lambda_{0,j}=0$, $j=1, ..., r_0$.

From (13) it follows that

$$\left\langle \sum_{j=1}^{r_0} \lambda_{0,j} L_{0,j}, z^{nr_0+p} \right\rangle + \left\langle \sum_{k \neq 0} \sum_{j=1}^{r_k} \lambda_{k,j} L_{k,j}, z^{nr_0+p} \right\rangle + \left\langle \sum_{j=1}^{M} \lambda_k^* L_k^*, z^{nr_0+p} \right\rangle = 0, \qquad p=0, ..., r_0-1, \quad n=0, 1, .... \tag{14}$$

Now we write $w(z)=z^{\alpha_0}\exp(-1/z^{r_0})\, g(z)$. Since $\beta_{0,1}, ..., \beta_{0,r_0}$, are the $r_0$-roots of the unity, we have

$$\begin{aligned}\left\langle \sum_{j=1}^{r_0} \lambda_{0,j} L_{0,j}, z^{nr_0+p} \right\rangle &= \sum_{j=1}^{r_0} \lambda_{0,j} \int_{\Gamma_{0,j}} z^{nr_0+p+\alpha_0} \exp\left(\frac{-1}{z^{r_0}}\right) g(z)\, dz \\ &= \int_{\Gamma_{0,1}} z^{nr_0+p+\alpha_0} \exp\left(\frac{-1}{t}\right) \sum_{j=1}^{r_0} \lambda_{0,j} \beta_{0,j}^{p+\alpha_0+1} g(z\beta_{0,j})\, dz\end{aligned}$$

and, letting $z^{r_0}=t$, the above equation reduces to

$$\frac{1}{r_0} \int_{\Gamma} t^n \exp\left(\frac{-1}{t}\right) t^{(p+\alpha_0+1-r_0)/r_0} \sum_{j=1}^{r_0} \lambda_{0,j} \beta_{0,j}^{\alpha_0+p+1} g(t^{1/r_0}\beta_{0,j})\, dt,$$

where $\Gamma$ is the path in Fig. 2. On the other hand, after making the substitution $z^{r_0}=t$, we also have

$$\begin{aligned}\left\langle \sum_{k \neq 0} \sum_{j=1}^{r_k} \lambda_{k,j} L_{k,j}, z^{nr_0+p} \right\rangle &= \frac{1}{r_0} \sum_{k \neq 0} \sum_{j=1}^{r_k} \lambda_{k,j} \int_{\bar{\Gamma}_{k,j}} t^n \exp\left(\frac{-1}{t}\right) t^{(p+\alpha_0-r_0+1)/r_0} g(t^{1/r_0})\, dt, \\ &\qquad p=0, ..., r_0-1, \quad n=0, 1, ...,\end{aligned}$$

where $\bar{\Gamma}_{k,j}$ is a curve in the region $|t|>l_0^{r_0}$. Furthermore

$$\begin{aligned}\left\langle \sum_{k=1}^{M} \lambda_k^* L_k^*, z^{nr_0+p} \right\rangle &= \frac{1}{r_0} \sum_{k=1}^{M} \lambda_k^* \int_{\bar{E}_k} t^n \exp\left(\frac{-1}{t}\right) t^{(p+\alpha_0-r_0+1)/r_0} g(t^{1/r_0})\, dt, \\ &\qquad p=0, ..., r_0-1, \quad n=0, 1, ...,\end{aligned}$$

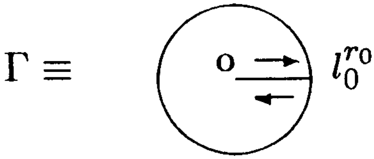


FIGURE 2

where $\bar{E}_k$ is a curve such that its part near zero is in the region $\Re(t) > 0$, and the corresponding integral converges.

Hence, from (14) it follows that

$$\int_\Gamma \frac{1}{t-\zeta} \exp\left(\frac{-1}{t}\right) t^{(p+\alpha-r_0+1)/r_0} \sum_{j=1}^{r_0} \lambda_{0,j} \beta_{0,j}^{\alpha_0+p+1} g(t^{1/r_0}\beta_{0,j})\, dt$$
$$+ \sum_{k\neq 0} \sum_{j=1}^{r_k} \lambda_{k,j} \int_{\bar{\Gamma}_{k,j}} \frac{1}{t-\zeta} \exp\left(\frac{-1}{t}\right) t^{(p+\alpha_0-r_0+1)/r_0} g(t^{1/r_0})\, dt$$
$$+ \sum_{k=1}^{M} \lambda_k^* \int_{\bar{E}_k} \frac{1}{t-\zeta} \exp\left(\frac{-1}{t}\right) t^{(p+\alpha_0-r_0+1)/r_0} g(t^{1/r_0})\, dt = 0$$

for $\zeta$ such that $|\zeta|$ is large enough and for each $p = 0, ..., r_0 - 1$. With $p$ fixed and denoting each term in the last expression in $H_1(\zeta)$, $H_2(\zeta)$, and $H_3(\zeta)$, it becomes

$$H_1(\zeta) + H_2(\zeta) + H_3(\zeta) = 0 \qquad \text{for } |\zeta| \text{ sufficiently large.}$$

For $\varepsilon > 0$, if we use $H_{1,\varepsilon}(\zeta)$ to refer to the integral of the function which defines $H_1(\zeta)$ but now over the path $\Gamma_\varepsilon$ of Fig. 3, we have $H_1(\zeta) = H_{1,\varepsilon}(\zeta)$ when $|\zeta| > l_0^{r_0}$, whence

$$H_{1,\varepsilon}(\zeta) + H_2(\zeta) + H_3(\zeta) = 0$$

for $|\zeta| > \varepsilon$ and $\zeta$ outside the curves $\bar{\Gamma}_{k,j}$ and $\bar{E}_k$.

Let $C$ be the curve in Fig. 4 and let $\zeta$ be a point such that $\varepsilon < |\zeta| < l_0^{r_0}$ and $\zeta \notin (\varepsilon, l_0^{r_0})$. By Cauchy's theorem we have

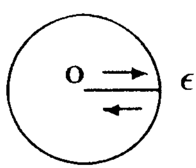


FIGURE 3

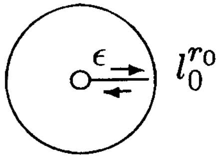


FIGURE 4

$$2\pi i \exp\left(\frac{-1}{\zeta}\right) \zeta^{(p+\alpha_0-r_0+1)/r_0} \sum_{j=1}^{r_0} \lambda_{0,j} \beta_{0,j}^{\alpha_0+p+1} g(\zeta^{1/r_0}\beta_{0,j})$$
$$= \int_C \frac{1}{t-\zeta} \exp\left(\frac{-1}{t}\right) t^{(p+\alpha_0-r_0+1)/r_0} \sum_{j=1}^{r_0} \lambda_{0,j} \beta_{0,j}^{\alpha_0+p+1} g(t^{1/r_0}\beta_{0,j})\, dt$$
$$= H_1(\zeta) - H_{1,\varepsilon}(\zeta) = H_1(\zeta) + H_2(\zeta) + H_3(\zeta). \tag{15}$$

Let $\zeta$ be a point in $(-\frac{1}{2}, 0)$. Then $|t-\zeta| \geqslant t$ when $t \in [0, l_0^{r_0}]$, and $|t-\zeta| \geqslant \frac{1}{2}$ when $t$ lies in $|t| = l_0^{r_0}$. Therefore $H_1(\zeta)$ is a bounded function when $\zeta \in (-\frac{1}{2}, 0)$. Moreover, $H_2(\zeta)$ and $H_3(\zeta)$ are bounded too in the same region. As a consequence, equality (15) only holds when

$$\sum_{j=1}^{r_0} \lambda_{0,j} \beta_{0,j}^{\alpha_0+p+1} g(\zeta^{1/r_0}\beta_{0,j}) = 0, \qquad p = 0, ..., r_0 - 1$$

from which $\lambda_{0,j} = 0$, $j = 1, ..., r_0$.

It is clear that the preceding work can be carried over to any multiple zero $a_k$ by a change $z - a_k = t$. Hence $\lambda_{k,j} = 0$, $j = 1, ..., r_k$, for every $k$ such that $a_k$ is a multiple zero. It remains to be proved that $\lambda_k^* = 0$ for $k = 1, ..., M$.

$$\left\langle \sum_{k=1}^{M} \lambda_k^* L_k^*, p \right\rangle = \sum_{k=1}^{M} \lambda_k^* \int_{E_k} w(z)\, p(z)\, dz = \int_X w(z) \sum_{k=1}^{M} \lambda_k^* \chi_{E_k}(z)\, p(z)\, dz,$$

where $X = \bigcup_{k=1}^{M} E_k$, and $\chi_{E_k}(z) = 1$ when $z \in E_k$ and zero otherwise. It is clear that this is a bounded functional over the space of continuous functions on $X$ and, since the complement of $X$ is connected and its interior is empty, by Merguelian's theorem, there exists only one extension of the functional over the continuous functions on $X$. Then, if

$$\left\langle \sum_{k=1}^{M} \lambda_k^* L_k^*, p \right\rangle = 0, \qquad \text{for every polynomial } p,$$

it is zero on the continuous functions. Hence, from Riesz Representation Theorem, this functional may be represented in a unique form and it follows that

$$w(z) \sum_{k=1}^{M} \lambda_k^* \chi_{E_k}(z) = 0$$

for every $z$ where this is a continuous function. Then $\lambda_k^* = 0$ for $k = 1, ..., M$.

## 3. REGULARIZATION

When $\Re(\alpha_k) \leqslant -1$ for some simple zero $a_k$, the corresponding integrals are not convergent and a regularization is needed. It will be done recurrently over the integer part of $\Re(\alpha_k)$.

Given the equation $D(\phi L) + \psi L = 0$, if $a$ is a zero of $\phi$ we denote

$$\phi(x) = (x-a)\, \phi_a(x), \qquad \psi(x) = (x-a)\, \psi_a(x) + \psi(a),$$

and, using Maroni's techniques, we consider

$$\langle (x-a)^{-1} L, p \rangle = \left\langle L, \frac{p(x) - p(a)}{x-a} \right\rangle.$$

PROPOSITION 3.1. *Let $a$ be one zero of $\phi$ such that $\psi(a) \neq 0$. If $\{L_1, ..., L_{s+1}\}$ is an FSS of the equation of class $s$ and type* (B)

$$D(\phi L) + (\psi - \phi_a) L = 0,$$

*then*

$$\{(x-a)^{-1} L_1 + M_1 \delta(x-a), ..., (x-a)^{-1} L_{s+1} + M_{s+1} \delta(x-a)\},$$

*where $M_j = -\langle L_j, \psi_a \rangle / \psi(a)$, is an FSS of $D(\phi L) + \psi L = 0$.*

*Proof.* Let $L_j^* = (x-a)^{-1} L_j + M_j \delta(x-a)$, then $L_j = (x-a) L_j^*$. Furthermore

$$D((x-a)^2\, \phi_a L_j^*) = D((x-a)\, \phi_a L_j) = -(\psi - \phi_a)\, L_j$$

and thus

$$D((x-a)^2\, \phi_a L_j^*) + (x-a)(\psi - \phi_a)\, L_j^* = 0.$$

Taking the derivative, we obtain $(x-a)(D(\phi L_j^*)+\psi L_j^*)=0$, which means that

$$D(\phi L_j^*)+\psi L_j^*=\langle L_j^*, \psi\rangle\, \delta(x-a).$$

Moreover

$$\begin{aligned}\langle L_j^*, \psi\rangle &= \left\langle L_j, \frac{\psi(x)-\psi(a)}{x-a}\right\rangle + M_j\langle \delta(x-a), \psi\rangle \\ &= \langle L_j, \psi_a\rangle + M_j\psi(a)=0\end{aligned}$$

from the definition of $M_j$, and it follows that $D(\phi L_j^*)+\psi L_j^*=0$.

On the other hand

$$\begin{vmatrix} \langle L_1^*, 1\rangle, & \cdot & , \langle L_{s+1}^*, 1\rangle \\ \langle L_1^*, x-a\rangle, & \cdot & , \langle L_{s+1}^*, x-a\rangle \\ \vdots & \vdots & \vdots \\ \langle L_1^*, (x-a)^s\rangle, & \cdot & , \langle L_{s+1}^*, (x-a)^s\rangle \end{vmatrix}$$

$$= \begin{vmatrix} M_1, & \cdot & , M_{s+1} \\ \langle L_1, 1\rangle, & \cdot & , \langle L_{s+1}, 1\rangle \\ \vdots & \vdots & \vdots \\ \langle L_1, (x-a)^{s-1}\rangle, & \cdot & , \langle L_{s+1}, (x-a)^{s-1}\rangle \end{vmatrix}$$

$$= -\frac{K}{\psi(a)} \begin{vmatrix} \langle L_1, (x-a)^s\rangle, & \cdot & , \langle L_{s+1}, (x-a)^s\rangle \\ \langle L_1, 1\rangle, & \cdot & , \langle L_{s+1}, 1\rangle \\ \vdots & \vdots & \vdots \\ \langle L_1, (x-a)^{s-1}\rangle, & \cdot & , \langle L_{s+1}, (x-a)^{s-1}\rangle \end{vmatrix},$$

where $K$ is the coefficient of degree $s+1$ of $\psi$ which is non-zero because the equation is of type (B). The last identity holds because the remaining terms in the first row are a linear combination of the others. By hypothesis $\{L_1, ..., L_{s+1}\}$ is an FSS and, from Lemma 2.1, the last determinant is non-zero. This means that $\{L_1^*, ..., L_{s+1}^*\}$ is an FSS of $D(\phi L)+\psi L=0$. Let us now explain the recursive process.

Suppose initially that only one $\alpha_k$ corresponding to a simple zero $a_k$ has $\Re(\alpha_k)\leqslant -1$ and that $\alpha_k \neq -1, -2, ...$. If $-2<\Re(\alpha_k)\leqslant -1$, equation $D(\phi L)+(\psi-\phi_{a_k})\, L=0$ is covered by Theorem 2.1 because

$$\frac{w'(z)}{w(z)} = -\frac{\psi(z)-\phi_{a_k}(z)+\phi'(z)}{\phi(z)} = -\frac{\psi(z)+\phi'(z)}{\phi(z)}+\frac{1}{z-a_k}.$$

By applying Proposition 3.1 we obtain the solution for $D(\phi L)+\psi L=0$. In order to apply Proposition 3.1 we need $\psi(a_k)\neq 0$, but this is equivalent to the condition $\alpha_k\neq -1$ because $a_k$ is a simple zero. By repeating the above process as many times as required by the integer part of $\Re(\alpha_k)$, we have the solution for case $\Re(\alpha_k)\leqslant -1$ provided that $\alpha_k\neq -1, -2, ...$.

Let us now solve case $\alpha_k=-1$, the solution of which can be extended with Proposition 3.1 to obtain the solution for $\alpha_k=-2, -3, ...$.

PROPOSITION 3.2. *Given the equation $D(\phi L)+\psi L=0$, suppose that, for some zero $a$ of $\phi$, $\psi(a)=0$. Let $\{L_1, ..., L_s\}$ be an FSS of the equation of class $s-1$, $D(\phi_a L)+\psi_a L=0$. Then*

$$\{\delta(x-a), (x-a)^{-1} L_1, ..., (x-a)^{-1} L_s\}$$

*is an FSS of $D(\phi L)+\psi L=0$.*

*Proof.* It is straightforward to show that $\delta(x-a)$ is a solution. Let $L_j^*=(x-a)^{-1} L_j$. Then $L_j=(x-a) L_j^*$, and it follows that

$$D((x-a)\,\phi_a L_j^*)=D(\phi_a L_j)=-\psi_a L_j=-(x-a)\,\psi_a L_j^*=-\psi L_j^*.$$

Moreover

$$\begin{vmatrix} \langle \delta(x-a), 1\rangle, & \cdot & \langle \delta(x-a), (x-a)^s\rangle \\ \langle L_1^*, 1\rangle, & \cdot & \langle L_1^*, (x-a)^s\rangle \\ \vdots & \vdots & \vdots \\ \langle L_s^*, 1\rangle, & \cdot & \langle L_s^*, (x-a)^s\rangle \end{vmatrix}$$

$$=\begin{vmatrix} 1 & 0 & \cdot & 0 \\ 0 & \langle L_1, 1\rangle & \cdot & \langle L_1, (x-a)^{s-1}\rangle \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \langle L_s, 1\rangle & \cdot & \langle L_s, (x-a)^{s-1}\rangle \end{vmatrix}$$

which is non-zero because $\{L_1, ..., L_s\}$ is an FSS. Hence $\{\delta(x-a), (x-a)^{-1} L_1, ..., (x-a)^{-1} L_s\}$ is an FSS of $D(\phi L)+\psi L=0$.

The equation $D(\phi_a L)+\psi_a L=0$, when regularization is necessary for only one zero, gives rise to two possibilities:

(1) The solution of the equation is covered by Theorem 2.1

(2) The equation reduces to $DL+KL=0$ where $K\neq 0$ is a constant.

The equation in case (2) yields

$$\begin{cases} K\langle L, 1\rangle = 0 \\ -n\langle L, x^{n-1}\rangle + K\langle L, x^n\rangle = 0, \qquad n \geqslant 1, \end{cases}$$

and $\langle L, x^n\rangle = 0, n \geqslant 0$, so $L = 0$.

Thus, the equation is solved when only one zero has its real part less than or equal to $-1$. If there were more than one zero with the real part less than or equal to $-1$, Propositions 3.1 and 3.2 could be used to reduce this case to the previous case.

EXAMPLE. We present an example of the regularization method for a semiclassical functional of class $s = N-1$ which covers Laguerre functionals ($N=1$), studied by Morton and Krall in [16], and an example of Airy ($N=3$) and Freud functionals ($N=4$). Examples of Airy functionals may be seen in [12] and for Freud ones see, for example, [1].

Let $L$ be such that

$$D(xL) + (Nx^N - \alpha - 1)\, L = 0.$$

One of the solutions is, for $\Re\alpha > -1$,

$$\langle L^{(\alpha)}, p\rangle = \int_0^\infty x^\alpha e^{-x^N} p(x)\, dx.$$

Let us consider a real number $\varepsilon$ such that $-1 < \varepsilon < 0$. Our aim is to obtain the solution for $\alpha = \varepsilon - 1, \varepsilon - 2, ..., \varepsilon - n, ...$ . From Proposition 3.1, the corresponding solution of

$$D(xL) + (Nx^N - (\varepsilon - n) - 1)\, L = 0$$

can be written as

$$L^{(\varepsilon - n)} = x^{-1} L^{(\varepsilon - n + 1)} + M_n\, \delta,$$

where

$$M_n = \frac{\langle L^{(\varepsilon - n + 1)}, Nx^{N-1}\rangle}{\varepsilon + 1 - n}.$$

With the same notation and using induction we get

$$L^{(\varepsilon - n)} = x^{-n} L^{(\varepsilon)} + \sum_{k=1}^{n} \frac{(-1)^{n-k}}{(n-k)!} M_k\, \delta^{(n-k)}, \tag{16}$$

where

$$\langle x^{-n}L^{(\varepsilon)}, p\rangle = \left\langle L^{(\varepsilon)}, \frac{1}{x^n}\left(p(x) - \sum_{j=0}^{n-1} \frac{p^{(j)}(0)}{j!}\, x^j\right)\right\rangle$$

$$= \int_0^\infty x^{\varepsilon - n}\, e^{-x^N} \left\{p(x) - \sum_{j=0}^{n-1} \frac{p^{(j)}(0)}{j!}\, x^j\right\} dx$$

and the derivatives of $\delta$ appear in (6) because $x^{-k}\delta = ((-1)^k/k!)\,\delta^{(k)}$. So, we have to obtain $M_k$, $k = 1, ..., n$.

Setting $k = vN + j$, $j = 1, ..., N$, $v = 1, 2, ...$, we have

$$M_{vN+j} = \frac{\langle L^{(\varepsilon - vN - j + 1)}, Nx^{N-1}\rangle}{\varepsilon + 1 - (vN + j)}$$

$$= \frac{N}{\varepsilon + 1 - (vN + j)} \langle x^{-1}L^{(\varepsilon - vN - j + 2)} + M_{vN+j}\,\delta, x^{N-1}\rangle$$

$$= \frac{N}{\varepsilon + 1 - (vN + j)} \langle L^{(\varepsilon - vN - j + N)}, 1\rangle$$

$$= \frac{N}{\varepsilon + 1 - (vN + j)} \langle x^{-1}L^{(\varepsilon - (v-1)N - j + 1)} + M_{(v-1)N+j}\,\delta, 1\rangle$$

$$= \frac{N}{\varepsilon + 1 - (vN + j)} M_{(v-1)N+j}.$$

As a consequence, for $j = 1, ..., N$ and $v = 0, 1, ...,$ we have

$$M_{vN+j} = \frac{N}{\varepsilon + 1 - (vN + j)}\, \frac{N}{\varepsilon + 1 - ((v-1)N + j)} \cdots \frac{N}{\varepsilon + 1 - j} \langle L^{(\varepsilon)}, x^{N-j}\rangle \tag{17}$$

because

$$M_j = \frac{N}{\varepsilon + 1 - j} \langle L^{(\varepsilon)}, x^{N-j}\rangle, \qquad j = 1, ..., N.$$

Hence, letting $M_{vN+j} = A_{vN+j}\langle L^{(\varepsilon)}, x^{N-j}\rangle$, and $n = kN + r$, $r = 1, ..., N$, expression (17) in (16) yields

$$\langle L^{(\varepsilon - kN - r)}, p\rangle$$

$$= \langle x^{-(kN+r)}L^{(\varepsilon)}, p\rangle + \sum_{v=0}^{k-1} \sum_{j=1}^{N} \frac{A_{vN+j}\, p^{((k-v)N + r - j)}(0)}{((k-v)N + r - j)!} \langle L^{(\varepsilon)}, x^{N-j}\rangle$$

$$+ \sum_{j=1}^{r} \frac{A_{kN+j}\, p^{(r-j)}(0)}{(r-j)!} \langle L^{(\varepsilon)}, x^{N-j}\rangle.$$

Then

$$\langle L^{(\varepsilon-kN-r)}, p\rangle = \int_0^\infty x^{\varepsilon-kN-r} e^{-x^N} \left\{ p(x) - \sum_{j=0}^{kN+r-1} \frac{p^{(j)}(0)}{j!} x^j + \sum_{v=0}^{k-1} \sum_{j=1}^{N} \frac{A_{vN+j} p^{(k-v)N+r-j}(0)}{((k-v)N+r-j)!} x^{(k+1)N+r-j} + \sum_{j=1}^{r} \frac{A_{kN+j} p^{(r-j)}(0)}{(r-j)!} x^{(k+1)N+r-j} \right\} dx$$

and the $A_{vN+j}$ are defined by the recurrent relation

$$A_j = \frac{N}{\varepsilon+1-j}, \qquad j=1, ..., N$$

$$A_{vN+j} = \frac{N}{\varepsilon+1-(vN+j)} A_{(v-1)N+j}, \qquad j=1, ..., N; \; v=1, 2, ... \quad \blacksquare$$

In cases $\alpha = -1, -2, ...$, the solutions have a different form. For $\alpha = -1$, $D(xL) + Nx^N L = 0$, by Proposition 3.2 one must solve

$$DL + Nx^{N-1} L = 0.$$

Any solution of this equation has the form

$$\langle L, p\rangle = \sum_{i=1}^{N-1} \lambda_i \int_{\gamma_i} p(z)\, e^{-z^N}\, dz,$$

where the $\gamma_i$ are defined in Theorem 2.1. Therefore, by Proposition 3.2, any solution $L_N^{(-1)}$ of $D(xL) + Nx^N L = 0$ may be written as

$$\langle L_N^{(-1)}, p\rangle = \sum_{i=1}^{N-1} \lambda_i \int_{\gamma_i} \frac{p(z)-p(0)}{z}\, e^{-z^N}\, dz + \lambda_N p(0)$$

and the corresponding solutions for $\alpha = -2, -3, ...$ must be obtained with Proposition 3.1 again. Letting

$$\langle L_{N,i}^{(-1)}, p\rangle = \int_{\gamma_i} \frac{p(z)-p(0)}{z}\, e^{-z^N}\, dz, \qquad i=1, ..., N-1,$$

$$\langle L_{N,N}^{(-1)}, p\rangle = \langle \delta, p\rangle,$$

in the same way as before, for $i=1, ..., N$ we have

$$\langle L_{N,i}^{(-1-kN-r)}, p\rangle = \langle x^{-(kN+r)} L_{N,i}^{(-1)}, p\rangle + \sum_{v=0}^{k-1} \sum_{j=1}^{N} \frac{A_{vN+j} p^{((k-v)N+r-j)}(0)}{((k-v)N+r-j)!} \langle L_{N,i}^{(-1)}, x^{N-j}\rangle + \sum_{j=1}^{r} \frac{A_{kN+j} p^{(r-j)}(0)}{(r-j)!} \langle L_{N,i}^{(-1)}, x^{N-j}\rangle,$$

where

$$A_j = -\frac{N}{j}, \qquad j=1, ..., N$$

$$A_{vN+j} = -\frac{N}{vN+j} A_{(v-1)N+j}, \qquad j=1, ..., N; \ v=1, 2, ... . \quad \blacksquare$$

*Remark.* Freud weights are explicitly related to this problem. In fact, the associated linear functional is a B-functional. The distributional equation allows us to obtain the nonlinear equations (the so-called Freud equations) of the coefficients of the three-term recurrence relation of the corresponding sequence of orthogonal polynomials. Furthermore, the solutions of such equations are given in terms of Hankel determinants whose entries are the moments $(\mu_k)$. They satisfy a linear recurrence relation as we pointed out in the Introduction. In a private communication, A. P. Magnus announced the connection between Freud equations and discrete Painlevé equations when $\phi=1$ and $\psi$ is an odd polynomial.

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